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Vibration of Thin Circular Rings

Part I. Solutions for Modal Characteristics and Forced Excitation

T. E. Lang

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NOMENCLATURE

a radius of ring, in.

A cross-sectional area of ring, in.^{2}

A_{n_i} constant of modal functions

b_n root parameter [Eq. (32)]

B_{n_i} constant of modal functions

c_n root parameter [Eq. (32)]

C_{n_i} constant of transient solution

d_n root parameter [Eq. (32)]

e_n root parameter [Eq. (32)]

E Young's modulus, lb/in.^{2}

f_n root parameter [Eq. (32)]

F_R external force—radial, lb

F_T external force—tangential, lb

h thickness of ring, in.

i $\sqrt{-1}$

I area moment of inertia of ring cross section, in.^{4}

L_i differential operator

m_{n_i} root [Eq. (31)]

$M, M_\phi, \mathcal{M}$ in-plane bending moment, in.-lb.

n mode number

$N, N_\phi, \mathcal{N}$ circumferential normal force, lb

p geometric constant [Eq. (12)]

Q_ϕ transverse shear force, lb

$Q_n(t)$ generalized force

t time variable, sec

v tangential displacement, in.

v_0 initial displacement—tangential, in.

$\dot{v}_0$ initial velocity—tangential, in./sec

$\mathcal{V}, \mathcal{V}_n$ characteristic or modal function—tangential

w radial displacement, in.

w_0 initial displacement—radial, in.

$\dot{w}_0$ initial velocity—radial, in./sec

$\mathcal{W}, \mathcal{W}_n$ characteristic or modal function—radial

x axial coordinate

z radial coordinate

α_n parameter [Eq. (37)]

β_n parameter [Eq. (46)]

γ_i ring solution [Eq. (64)]

δ_n root parameter [Eq. (32)]

ε_ϕ extensional strain

ζ_{n_i} root [Eq. (32)]

$\eta_n(t)$ generalized time variable

μ_n root parameter [Eq. (32)]

ν_n parameter [Eq. (47)]

$\xi_L^*, \xi_M^*, \xi_U^*$ change-over values of ξ_n

ξ_n frequency parameter [Eq. (30)]

ρ mass density, lb-sec^2/in.

τ time variable, sec

τ_n parameter [Eq. (51)]

ϕ angle coordinate, rad

ϕ^* central angle of incomplete ring, rad

χ_ϕ change of curvature, 1/in.

ψ_i ring solution [Eq. (64)]

ψ bending slope, rad

ω_0 natural frequency of pure extensional mode, rad/sec

ω, ω_n frequency, rad/sec

Subscripts

i

j

k

l

m

n

ϕ

ABSTRACT

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The inextensional and extensional deformation theories are applied to the analysis of the in-plane vibration of thin circular rings. Consideration is directed first to defining the range of validity of the two deformation theories in terms of mode number and thickness to radius ratios of complete circular rings. The eigenfunctions and procedure for finding eigenvalues for complete and incomplete rings are then established for both types of deformation theories. Numerical evaluation of the symmetric-modal and frequency characteristics of a complete ring clamped at one point on its circumference is performed for the inextensional case. Verification of the inextensional theory is established by comparing experimental results with the calculated results for the three lowest symmetric flexural modes. Finally, particular solutions of the extensional and inextensional ring equations are derived in integral form.

I. INTRODUCTION

For in-plane vibration of incomplete and complete thin circular rings, the equations of motion based upon inextension of the ring neutral axis have been used extensively to determine low mode properties. Lamb, in 1888, using the inextensional equations, determined the values of the fundamental symmetric and asymmetric natural frequencies for a shallow "free-free" ring segment (Ref. 1). Subsequent to Lamb's work, particular cases of constrained rings have been analyzed by others. The eigenfunctions and eigenfrequencies of an unconstrained complete ring were determined by Love, who recognized that the vibration of a complete ring is characterized by an integer number of wavelengths which leads to appreciable simplification of the modal functions and the frequency equation (Ref. 2). The "clamped-clamped" ring was considered in detail by Waltking (Ref. 3) with further work more recently reported by Archer (Ref. 4). Extension of the analysis of the "free-free" ring to a nonshallow geometry was considered by Morley in computing the first ten eigenvalues of the symmetric and asymmetric modes of the split ring (Ref. 5). The equations obtained by allowing the ring neutral axis to extend have not been used to any extent, although they have definite application when modal characteristics are to be defined in an intermediate range between valid inextensional theory and theory that accounts for shear deflection.

Apparently, from the lack of tabulated modal characteristics in the literature, forced-motion solutions for vibrating rings have been given little consideration. The only work known to this author is the article by Archer (Ref. 4), which considers the case of a "clamped-clamped" incomplete ring subjected to a periodic displacement input at one end.

In this Report, the methods for determining modal characteristics of incomplete and complete circular rings are reviewed, with additional consideration given to the orthogonality properties of the modal functions, limitations of the inextensional and extensional theories, and classification of the homogeneous boundary conditions for circular rings. Particular solutions are derived in general terms for both the extensional and inextensional deformation theories. The modal functions appearing in the particular solution of the inextensional case are cataloged in Part II[1] of this Report for rings having a central angle of 180 deg. With these data, forced-vibration solutions of half-rings, and from symmetry considerations, certain complete rings can be evaluated numerically.

[1] *Vibration of Thin Circular Rings, Part II. Modal Functions and Eigenvalues of Constrained Half-Rings* (to be published).

II. EQUATIONS OF MOTION

Cylindrical coordinates suitable for defining position on a ring are shown in Fig. 1. Corresponding respectively to the coordinates ϕ and z, displacements v and w are chosen that are positive in the direction of a positive coordinate increment.

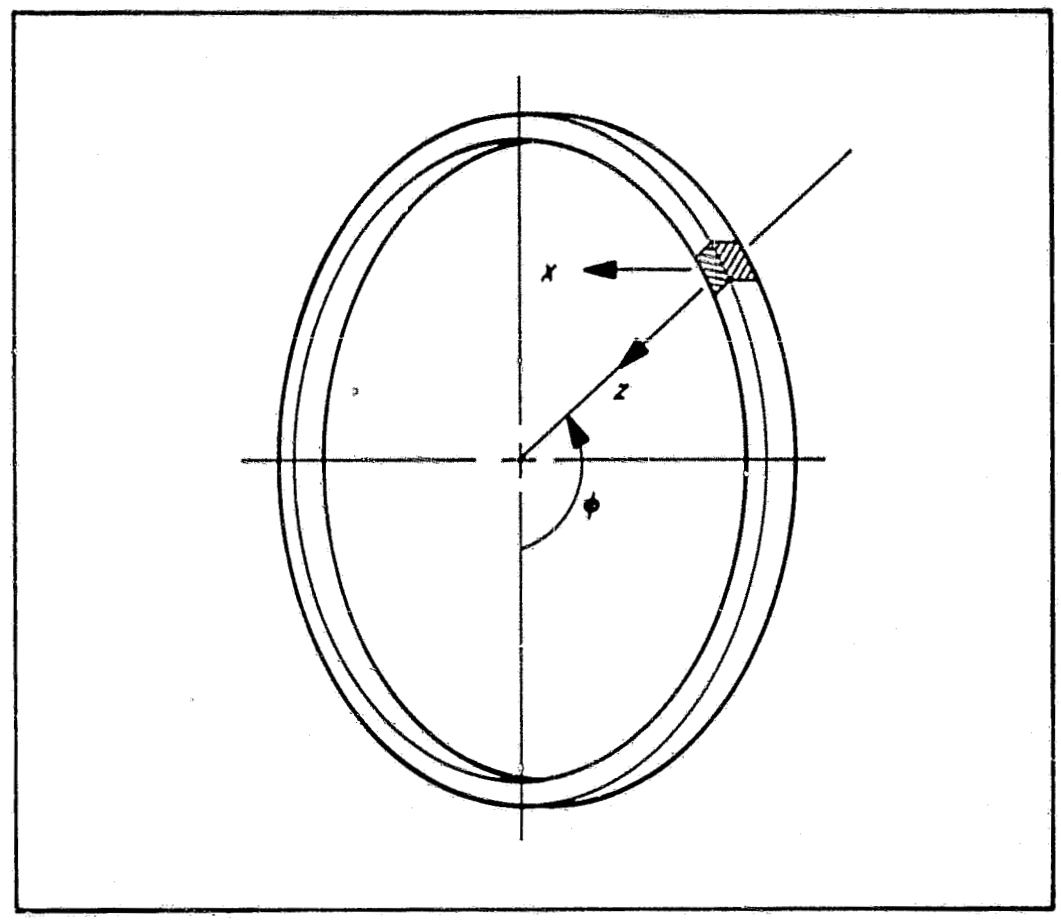

Fig. 1. Definition of coordinates

Differential equations that represent the in-plane vibrational motion of thin circular rings are derived from consideration of the variation of forces and moments across an element of the ring. Consider the dynamic equilibrium of a small element of the ring (shaded segment in Fig. 1). Acting on the element are the transverse shear force Q_ϕ, circumferential normal force N_ϕ, and bending moment M_ϕ, as shown in Fig. 2. Additional forces that act when the ring vibrates are the linear inertias and external forces, also shown in Fig. 2 (rotary inertia is neglected). By summing forces in the tangential and radial directions, and moments about the x axis, the following equations are obtained:

$$\frac{\partial N_\phi}{\partial \phi} - Q_\phi - \rho a A \frac{\partial^2 v}{\partial t^2} = -F_T(\phi, t) \tag{1}$$

$$N_\phi + \frac{\partial Q_\phi}{\partial \phi} - \rho a A \frac{\partial^2 w}{\partial t^2} = -F_R(\phi, t) \tag{2}$$

$$Q_\phi = -\frac{1}{a}\frac{\partial M_\phi}{\partial \phi} \tag{3}$$

The transverse shear force Q_ϕ can be eliminated from Eq. (1) and (2) by use of Eq. (3), yielding

$$\frac{\partial N}{\partial \phi} + \frac{1}{a}\frac{\partial M}{\partial \phi} - \rho a A \frac{\partial^2 v}{\partial t^2} = -F_T(\phi, t) \tag{4}$$

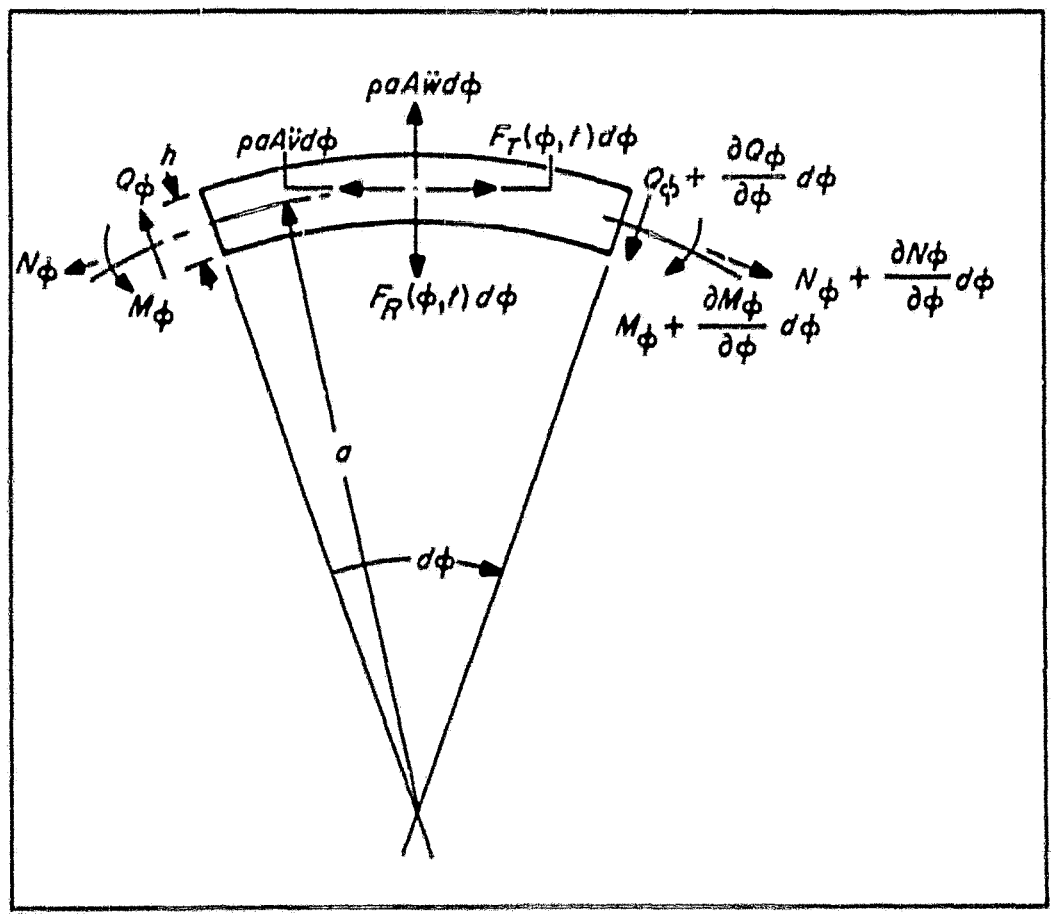

Fig. 2. Definition of forces and moments acting on a ring element

$$N - \frac{1}{a}\frac{\partial^2 M}{\partial \phi^2} - \rho a A \frac{\partial^2 w}{\partial t^2} = -F_R(\phi, t) \tag{5}$$

where the subscript ϕ has been dropped since it is the only space variable in the equations. In converting these force and moment equations to displacement equations, the elasticity assumptions associated with thin-shell theory will be employed. The first-order approximations of the required relations between the displacements v and w and the normal force and bending moment as derived by Timoshenko (Ref. 6) are

$$N_\phi = EA\varepsilon_\phi \tag{6}$$

$$M_\phi = EI\chi_\phi \tag{7}$$

where

$$\varepsilon_\phi = \frac{1}{a}\left(\frac{\partial v}{\partial \phi} - w\right) \tag{8}$$

$$\chi_\phi = \frac{1}{a^2}\left(\frac{\partial v}{\partial \phi} + \frac{\partial^2 w}{\partial \phi^2}\right) \tag{9}$$

Eq. (6)–(9) represent first-order approximations of the load-displacement relations for extensional deformation and flexure of a ring segment. The extensional strain ε_ϕ is assumed non-zero in defining the normal stress resultant N_ϕ. However, it is assumed that this same strain has little effect on the moment, which is dependent upon changes of curvature, and in deriving the moment-displacement relation the condition is used that $\varepsilon_\phi = 0$. With these approximations, reciprocity holds and the appropriate orthogonality expression of the ring mode shapes can be established. Dropping the ϕ subscript and substituting Eq. (6)–(9) into Eq. (4)–(5), the coupled displacement equations of motion are obtained as

$$(1+p)\frac{\partial^2 v}{\partial \phi^2} - \frac{\partial w}{\partial \phi} + p\frac{\partial^3 w}{\partial \phi^3} - \frac{1}{\omega_0^2}\frac{\partial^2 v}{\partial t^2} = -\frac{a}{EA}F_T(\phi, t) \tag{10}$$

and

$$\frac{\partial v}{\partial \phi} - w - p\frac{\partial^3 v}{\partial \phi^3} - p\frac{\partial^4 w}{\partial \phi^4} - \frac{1}{\omega_0^2}\frac{\partial^2 w}{\partial t^2} = -\frac{a}{EA}F_R(\phi, t) \tag{11}$$

where

$$p = \frac{I}{Aa^2} \tag{12}$$

$$\omega_0^2 = \frac{E}{\rho a^2} \tag{13}$$

In the analysis that follows, both homogeneous and particular solutions to Eq. (10) and (11) are derived.

III. RANGE OF VALIDITY OF THE RING EQUATIONS

The coupled homogeneous equations, written in operator notation, are

$$L_1 v - L_2 w = 0 \tag{14}$$

$$L_2 v - L_3 w = 0 \tag{15}$$

where

$$L_1 = (1 + p)\frac{\partial^2}{\partial \phi^2} - \frac{1}{\omega_0^2}\frac{\partial^2}{\partial t^2} \tag{16}$$

$$L_2 = \frac{\partial}{\partial \phi} - p\frac{\partial^3}{\partial \phi^3} \tag{17}$$

$$L_3 = 1 + p\frac{\partial^4}{\partial \phi^4} + \frac{1}{\omega_0^2}\frac{\partial^2}{\partial t^2} \tag{18}$$

By operator multiplication, uncoupled equations may be derived, namely,

$$(L_1 L_3 - L_2 L_2)\, v = 0 \tag{19}$$

$$(L_1 L_3 - L_2 L_2)\, w = 0 \tag{20}$$

When either of these equations is expanded, using Eq. (16)–(18), the resulting differential equation for v or w is of the form

$$\frac{\partial^6 w}{\partial \phi^6} + 2\frac{\partial^4 w}{\partial \phi^4} - \frac{1}{\omega_0^2}\frac{\partial^6 w}{\partial \phi^4 \partial t^2} + \frac{\partial^2 w}{\partial \phi^2} + \frac{1}{\omega_0^2}\frac{\partial^4 w}{\partial \phi^2 \partial t^2} + \frac{1}{p\omega_0^2}\frac{\partial^4 w}{\partial \phi^2 \partial t^2} - \frac{1}{p\omega_0^2}\frac{\partial^2 w}{\partial t^2} - \frac{1}{p\omega_0^4}\frac{\partial^4 w}{\partial t^4} = 0 \tag{21}$$

Assuming a solution

$$w(\phi, t) = \mathcal{W}(\phi)\, e^{i\omega t} \tag{22}$$

and substituting it into Eq. (21) yields

$$\frac{d^6 \mathcal{W}}{d\phi^6} + \left(2 + \frac{\omega^2}{\omega_0^2}\right)\frac{d^4 \mathcal{W}}{d\phi^4} + \left(1 - \frac{\omega^2}{\omega_0^2} - \frac{\omega^2}{p\omega_0^2}\right)\frac{d^2 \mathcal{W}}{d\phi^2} + \frac{\omega^2}{p\omega_0^2}\left(1 - \frac{\omega^2}{\omega_0^2}\right)\mathcal{W} = 0 \tag{23}$$

For thin rings, $p << 1$ so $\omega^2/p\omega_0^2 >> \omega^2/\omega_0^2$; hence, the coefficient of the second derivative term in Eq. (23) can be simplified immediately. Furthermore, if $\omega^2/\omega_0^2 << 1$, then Eq. (23), or similarly Eq. (21), can be reduced to the following:

$$\frac{\partial^6 w}{\partial \phi^6} + 2\frac{\partial^4 w}{\partial \phi^4} + \frac{\partial^2 w}{\partial \phi^2} + \frac{1}{p\omega_0^2}\frac{\partial^4 w}{\partial \phi^2 \partial t^2} - \frac{1}{p\omega_0^2}\frac{\partial^2 w}{\partial t^2} = 0 \tag{24}$$

which is the classical inextensional equation. Using the unconstrained complete-ring frequency equation (Ref. 2, p. 293), namely,

$$\frac{\omega^2}{p\omega_0^2} = \frac{n^2(n^2 - 1)^2}{n^2 + 1} \tag{25}$$

or for large n

$$\frac{\omega^2}{p\omega_0^2} \simeq n^4 \tag{26}$$

it is found that Eq. (24) is valid for $n \leq 4$, for $h/a = 0.10$; and for $n \leq 9$, for $h/a = 0.01$. Hence, the inextensional ring equation is valid for analyzing the modal characteristic or forced response of rings when modes lower than the fifth are important in relatively thick rings ($h/a = 0.10$) with larger ranges of modes being accurately represented for thinner rings. Furthermore, it is reasonable to assume that the results obtained for the free complete ring should apply also to rings having different end constraints if the wavelength for a mode shape of the constrained ring is approximately the same length as that for the free complete ring. This assures that, for comparable displacements, the changes of curvature are approximately the same in both cases.

Applying to the ring problem the beam theory criterion that shear deflection is not important for length-to-depth ratios greater than 10, it is found that shear deflection should be considered for modes corresponding to $n > 7$ for $h/a = 0.10$ and $n > 30$ for $h/a = 0.01$. These approximate numbers are based upon the relative importance of shear deflection compared to bending deflection in the static deflection of, say, a pinned-pinned beam. Hence, in the modal range between the inextensional theory and the theory including shear deflection, the extensional theory assumes importance which is characterized by Eq. (21) itself.

IV. THE EXTENSIONAL EQUATIONS

A. Solution of the Homogeneous Equations

As already demonstrated, the uncoupling of the homogeneous ring equations results in two equations in which a single operator acts on the dependent variables v and w [(see Section III, Eq. (19), (20)]; hence, by replacing w with v in Eq. (21), the differential equation in v is obtained. The equation is

$$\frac{\partial^6 v}{\partial \phi^6} + 2\frac{\partial^4 v}{\partial \phi^4} + \frac{\partial^2 v}{\partial \phi^2} - \frac{1}{\omega_0^2}\frac{\partial^6 v}{\partial \phi^4 \partial t^2} + \frac{1}{\omega_0^2}\frac{\partial^4 v}{\partial \phi^2 \partial t^2} + \frac{1}{\omega_0^2 p}\frac{\partial^4 v}{\partial \phi^2 \partial t^2} - \frac{1}{\omega_0^2 p}\frac{\partial^2 v}{\partial t^2} - \frac{1}{\omega_0^4 p}\frac{\partial^4 v}{\partial t^4} = 0 \qquad (27)$$

This equation is not separable; however, solutions sinusoidal in time may be assumed. Since eigensolutions are anticipated, the solution for, say, the nth mode may be assumed as follows:

$$v(\phi, t) = \mathcal{V}_n(\phi)\, e^{i\omega_n t} \qquad (28)$$

Hence, Eq. (27) becomes

$$\frac{d^6\mathcal{V}_n}{d\phi^6} + (2 + p\xi_n)\frac{d^4\mathcal{V}_n}{d\phi^4} + (1 - \xi_n - p\xi_n)\frac{d^2\mathcal{V}_n}{d\phi^2} + \xi_n(1 - p\xi_n)\mathcal{V}_n = 0 \qquad (29)$$

where

$$\xi_n = \frac{\omega_n^2}{p\omega_0^2} \qquad (30)$$

The quantities p and ω_0^2 are defined by Eq. (12) and (13). Since Eq. (29) is a linear differential equation with constant coefficients, a solution for $\mathcal{V}_n$ is

$$\mathcal{V}_n = \sum_{i=1}^{6} A_{n_i}\, e^{m_{n_i}\phi} \qquad (31)$$

which leads to the auxiliary equation in m_n, namely

$$m_n^6 + (2 + p\xi_n)m_n^4 + (1 - \xi_n - p\xi_n)m_n^2 + \xi_n(1 - p\xi_n) = 0$$

Letting $\zeta_n = m_n^2$

$$\zeta_n^3 + (2 + p\xi_n)\zeta_n^2 + (1 - \xi_n - p\xi_n)\zeta_n + \xi_n(1 - p\xi_n) = 0 \qquad (32)$$

The roots of this equation are of four types, depending upon the value of ξ_n. Designating $\xi_L^*, \xi_M^*, \xi_U^*$ as the lower, middle, and upper change-over values respectively, explicit forms of the roots may be defined. For $\xi_n < \xi_L^*$ the three roots of Eq. (32) are non-equal, negative, real numbers. For $\xi_L^* < \xi_n < \xi_M^*$, one root is negative real and the other two roots are complex conjugates. For $\xi_M^* < \xi_n < \xi_U^*$ one root is negative real and the others are positive real. Finally, for $\xi_n > \xi_U^*$ two roots are unequal negative real and one root is positive real. The four cases will be considered separately and solutions for $\mathcal{V}_n$ and $\mathcal{W}_n$ derived.

Case I: $\xi_n < \xi_L^*$

The roots of Eq. (32) are of the form

$$\zeta_{n_1} = -d_n^2$$
$$\zeta_{n_2} = -e_n^2$$
$$\zeta_{n_3} = -f_n^2$$

Therefore, the six roots m_{n_i} are

$$m_{n_1} = id_n$$
$$m_{n_2} = -id_n$$
$$m_{n_3} = ie_n$$
$$m_{n_4} = -ie_n \qquad (33)$$
$$m_{n_5} = if_n$$
$$m_{n_6} = -if_n$$

and the solution for $\mathcal{V}_n$ may be written immediately by substituting Eq. (33) into the expanded form of Eq. (31). The solution for $\mathcal{V}_n$, expressed in terms of trigonometric functions, is

$$\mathcal{V}_n = A_{n_1}\cos d_n\phi + A_{n_2}\sin d_n\phi + A_{n_3}\cos e_n\phi + A_{n_4}\sin e_n\phi + A_{n_5}\cos f_n\phi + A_{n_6}\sin f_n\phi \qquad (34)$$

The corresponding solution for $\mathcal{W}_n$ is

$$\mathcal{W}_n = B_{n_1}\cos d_n\phi + B_{n_2}\sin d_n\phi + B_{n_3}\cos e_n\phi + B_{n_4}\sin e_n\phi + B_{n_5}\cos f_n\phi + B_{n_6}\sin f_n\phi \qquad (35)$$

The solutions for $\mathcal{V}_n$ and $\mathcal{W}_n$ contain twelve unknowns: $A_{n_1} \cdots A_{n_6}$ and $B_{n_1} \cdots B_{n_6}$. However, for any particular ring configuration there are only six boundary conditions; therefore, six of the coefficients must be eliminated. This can be done by substituting Eq. (34) and (35) into one of the coupled homogeneous differential equations [Eq. (10) or (11)] and collecting terms of like trigonometric functions. The resulting relations between the A_{n_i} and B_{n_i} are

$$
\begin{aligned}
B_{n_1} &= -\alpha_{n_d} A_{n_2} \\
B_{n_2} &= \alpha_{n_d} A_{n_1} \\
B_{n_3} &= -\alpha_{n_e} A_{n_4} \\
B_{n_4} &= \alpha_{n_e} A_{n_3} \\
B_{n_5} &= -\alpha_{n_f} A_{n_6} \\
B_{n_6} &= \alpha_{n_f} A_{n_5}
\end{aligned} \tag{36}
$$

where

$$\alpha_{n_d} = \frac{p\xi_n - d_n^2(1+p)}{d_n(1+pd_n^2)} \tag{37}$$

$$\alpha_{n_e} = \frac{p\xi_n - e_n^2(1+p)}{e_n(1+pe_n^2)} \tag{38}$$

$$\alpha_{n_f} = \frac{p\xi_n - f_n^2(1+p)}{f_n(1+pf_n^2)} \tag{39}$$

Hence, the solution for $\mathcal{W}_n$ in terms of the six constants $A_{n_1} \cdots A_{n_6}$ is

$$
\begin{aligned}
\mathcal{W}_n = {} & A_{n_1}\alpha_{n_d} \sin d_n\phi - A_{n_2}\alpha_{n_d} \cos d_n\phi \\
& + A_{n_3}\alpha_{n_e} \sin e_n\phi - A_{n_4}\alpha_{n_e} \cos e_n\phi \\
& + A_{n_5}\alpha_{n_f} \sin f_n\phi - A_{n_6}\alpha_{n_f} \cos f_n\phi
\end{aligned} \tag{40}
$$

A special condition of Case I is when $\omega \to 0$, which is the condition for defining the rigid-body motion. Setting $\omega = 0$ in Eq. (32) and repeating the steps derived above, the general solutions for the rigid-body mode shapes, designated by $\mathcal{V}_0$ and $\mathcal{W}_0$, are

$$
\begin{aligned}
\mathcal{V}_0 = {} & A_1 + A_2\phi + A_3 \cos\phi + A_4 \sin\phi \\
& + A_5\phi \cos\phi + A_6\phi \sin\phi
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
\mathcal{W}_0 = {} & A_2 - A_3 \sin\phi + A_4 \cos\phi \\
& + A_5\left[\frac{1-p}{1+p}\cos\phi - \phi\sin\phi\right] \\
& + A_6\left[\frac{1-p}{1+p}\sin\phi + \phi\cos\phi\right]
\end{aligned} \tag{42}
$$

Case II: $\xi_L^* < \xi_n < \xi_M^*$

The roots of Eq. (32) are of the form

$$
\begin{aligned}
\zeta_{n_1} &= -d_n^2 \\
\zeta_{n_2} &= b_n + ic_n \\
\zeta_{n_3} &= b_n - ic_n
\end{aligned}
$$

Therefore, the six roots m_{n_i} are

$$
\begin{aligned}
m_{n_1} &= id_n \\
m_{n_2} &= -id_n \\
m_{n_3} &= \delta_n + i\mu_n \\
m_{n_4} &= \delta_n - i\mu_n \\
m_{n_5} &= -\delta_n - i\mu_n \\
m_{n_6} &= -\delta_n + i\mu_n
\end{aligned} \tag{43}
$$

and the solutions for $\mathcal{V}_n$ and $\mathcal{W}_n$, using the established procedure for eliminating the B_{n_i}, are

$$
\begin{aligned}
\mathcal{V}_n = {} & A_{n_1} \cos d_n\phi + A_{n_2} \sin d_n\phi \\
& + A_{n_3} \cos \mu_n\phi \cosh \delta_n\phi + A_{n_4} \sin \mu_n\phi \cosh \delta_n\phi \\
& + A_{n_5} \cos \mu_n\phi \sinh \delta_n\phi + A_{n_6} \sin \mu_n\phi \sinh \delta_n\phi
\end{aligned} \tag{44}
$$

and

$$
\begin{aligned}
\mathcal{W}_n = {} & A_{n_1}\alpha_{n_d} \sin d_n\phi - A_{n_2}\alpha_{n_2} \cos d_n\phi \\
& - A_{n_3}\left[\beta_n \sin \mu_n\phi \cosh \delta_n\phi + \nu_n \cos \mu_n\phi \sinh \delta_n\phi\right] \\
& + A_{n_4}\left[\beta_n \cos \mu_n\phi \cosh \delta_n\phi - \nu_n \sin \mu_n\phi \sinh \delta_n\phi\right] \\
& - A_{n_5}\left[\beta_n \sin \mu_n\phi \sinh \delta_n\phi + \nu_n \cos \mu_n\phi \cosh \delta_n\phi\right] \\
& + A_{n_6}\left[\beta_n \cos \mu_n\phi \sinh \delta_n\phi - \nu_n \sin \mu_n\phi \cosh \delta_n\phi\right]
\end{aligned} \tag{45}
$$

where

$$\beta_n = \frac{\mu_n\{(1+p)(\delta_n^2+\mu_n^2)[1+p(\delta_n^2+\mu_n^2)] - p\xi_n(1+p\mu_n^2-3p\delta_n^2)\}}{(\delta_n^2+\mu_n^2)[p^2(\delta_n^2+\mu_n^2)^2 - 2p(\delta_n^2-\mu_n^2)+1]} \tag{46}$$

$$\nu_n = \frac{\delta_n\{(1+p)(\delta_n^2+\mu_n^2)[-1+p(\delta_n^2+\mu_n^2)] - p\xi_n(1-p\delta_n^2+3p\mu_n^2)\}}{(\delta_n^2+\mu_n^2)[p^2(\delta_n^2+\mu_n^2)^2 - 2p(\delta_n^2-\mu_n^2)+1]} \tag{47}$$

Case III: $\xi_M^* < \xi_n < \xi_U^*$

The roots of Eq. (32) are of the form

$$
\begin{aligned}
\zeta_{n_1} &= -d_n^2 \\
\zeta_{n_2} &= e_n^2 \\
\zeta_{n_3} &= f_n^2
\end{aligned}
$$

which result in modal functions

$$\mathcal{V}_n = A_{n_1}\cos d_n\phi + A_{n_2}\sin d_n\phi + A_{n_3}\cosh e_n\phi + A_{n_4}\sinh e_n\phi + A_{n_5}\cosh f_n\phi + A_{n_6}\sinh f_n\phi \quad (48)$$

$$\mathcal{W}_n = A_{n_1}\alpha_{n_d}\sin d_n\phi - A_{n_2}\alpha_{n_d}\cos d_n\phi + A_{n_3}\tau_{n_e}\sinh e_n\phi + A_{n_4}\tau_{n_e}\cosh e_n\phi + A_{n_5}\tau_{n_f}\sinh f_n\phi + A_{n_6}\tau_{n_f}\cosh f_n\phi \quad (49)$$

where

$$\tau_{n_e} = \frac{p\xi_n + e_n^2(1+p)}{e_n(1-pe_n^2)}$$

$$\tau_{n_f} = \frac{p\xi_n + f_n^2(1+p)}{f_n(1-pf_n^2)} \quad (50)$$

Case IV: $\xi_n > \xi_t^*$

The roots of Eq. (32) are

$$\zeta_{n_1} = -d_n^2$$

$$\zeta_{n_2} = -e_n^2$$

$$\zeta_{n_3} = f_n^2$$

which lead to the following modal functions:

$$\mathcal{V}_n = A_{n_1}\cos d_n\phi + A_{n_2}\sin d_n\phi + A_{n_3}\cos e_n\phi + A_{n_4}\sin e_n\phi + A_{n_5}\cosh f_n\phi + A_{n_6}\sinh f_n\phi \quad (51)$$

$$\mathcal{W}_n = A_{n_1}\alpha_{n_d}\sin d_n\phi - A_{n_2}\alpha_{n_d}\cos d_n\phi + A_{n_3}\alpha_{n_e}\sin e_n\phi - A_{n_4}\alpha_{n_e}\cos e_n\phi + A_{n_5}\tau_{n_f}\sinh f_n\phi + A_{n_6}\tau_{n_f}\cosh f_n\phi \quad (52)$$

where α_{n_d}, α_{n_e}, and τ_{n_f} are defined by Eq. (37), (38), and (50) respectively.

The modal functions $\mathcal{V}_n$ and $\mathcal{W}_n$ that have been derived are general expressions, since the constants $A_{n_1} \cdots A_{n_6}$ and ξ_n have not been determined. To proceed further, it is necessary to define a particular ring configuration so that definite boundary conditions can be specified. Once this is done, six simultaneous equations involving the six constants $A_{n_1} \cdots A_{n_6}$ are obtained by substituting the expressions for $\mathcal{V}_n$ and $\mathcal{W}_n$ into the statements of the boundary conditions. Since these equations are homogeneous, five of the A_{n_i} must be expressed in terms of the sixth, this sixth constant appearing as a normalizing factor in front of the modal functions. This procedure is identical to that employed in beam theory; however, the algebraic manipulations are more tedious.

If a matrix is formed from the coefficients of the A_{n_i} in the six simultaneous equations, then the determinant of this matrix will equal zero for certain values of ξ_n, which are the eigenvalues for the particular ring configuration being studied. In beam theory, the determinant of the frequency matrix is a simple transcendental equation of two functions for which the eigenvalues may readily be determined by plotting the functions and locating intersecting points. However, in the ring problem, the transcendental equation is no longer simple, so that an alternate approach may be taken to determine the eigenvalues. The procedure involves evaluating the terms in the frequency matrix for a specific value of ξ_n. Then the value of the determinant is computed. After repeating these operations for several values of ξ_n, a plot of the determinant values vs ξ_n is made and the values of ξ_n where the curve crosses the zero-determinant axis are the eigenvalues. To clarify the procedure described above for determining the normal functions and eigenfrequencies of a specific ring configuration a numerical example should be considered. However, because the modal functions that have been derived based upon the extensional equations are algebraically very complex, an example using these equations will not be attempted.

B. Orthogonality of the Ring Modes

An orthogonality condition for the ring modal functions is derived by operating on the ring equations so that a symmetric form of Green's formula is obtained. To accomplish this, modified forms of Eq. (4)–(7) are used in which the external forces are dropped. The equations are

$$\frac{\partial N}{\partial\phi} + \frac{1}{a}\frac{\partial M}{\partial\phi} = \rho a A\frac{\partial^2 v}{\partial t^2} \quad (53)$$

$$N - \frac{1}{a}\frac{\partial^2 M}{\partial\phi^2} = \rho a A\frac{\partial^2 w}{\partial t^2} \quad (54)$$

$$\frac{a}{EI}M - \frac{\partial v}{\partial\phi} - \frac{\partial^2 w}{\partial\phi^2} = 0 \quad (55)$$

$$\frac{a}{EA}N - \frac{\partial v}{\partial\phi} + w = 0 \quad (56)$$

By assuming solutions for v, w, M, and N of the form

$$\begin{aligned} v(\phi, t) &= \mathcal{V}(\phi)\, e^{i\omega t} \\ w(\phi, t) &= \mathcal{W}(\phi)\, e^{i\omega t} \\ M(\phi, t) &= \mathcal{M}(\phi)\, e^{i\omega t} \\ N(\phi, t) &= \mathcal{N}(\phi)\, e^{i\omega t} \end{aligned} \tag{57}$$

and substituting these solutions into Eq. (53)–(56), the resulting differential equations will be in terms of four dependent variables $\mathcal{V}$, $\mathcal{W}$, $\mathcal{M}$, $\mathcal{N}$, and one independent variable ϕ. If one set of solutions $\mathcal{V}_1$, $\mathcal{W}_1$, $\mathcal{M}_1$, $\mathcal{N}_1$ is assumed, then the equations are

$$\frac{d\mathcal{N}_1}{d\phi} + \frac{1}{a}\frac{d\mathcal{M}_1}{d\phi} = -\rho a A \omega_1^2 \mathcal{V}_1 \tag{58}$$

$$\mathcal{N}_1 - \frac{1}{a}\frac{d^2\mathcal{M}_1}{d\phi^2} = -\rho a A \omega_1^2 \mathcal{W}_1 \tag{59}$$

$$\frac{a^2}{EI}\mathcal{M}_1 - \frac{d\mathcal{V}_1}{d\phi} - \frac{d^2\mathcal{W}_1}{d\phi^2} = 0 \tag{60}$$

$$\frac{a}{EA}\mathcal{N}_1 - \frac{d\mathcal{V}_1}{d\phi} + \mathcal{W}_1 = 0 \tag{61}$$

Assuming a second set of solutions $\mathcal{V}_2$, $\mathcal{W}_2$, $\mathcal{M}_2$, $\mathcal{N}_2$, corresponding to another mode, Eq. (58)–(61) are multiplied by these quantities in respective order. Then, by multiplying the resulting equations by $d\phi$ and integrating over the arc length of the ring and adding, the following equation is obtained:

$$\begin{aligned} \int_0^{\phi^*} \Bigg\{ \mathcal{V}_2 \left(\frac{d\mathcal{N}_1}{d\phi} + \frac{1}{a}\frac{d\mathcal{M}_1}{d\phi} \right) + \mathcal{W}_2 \left(\mathcal{N}_1 - \frac{1}{a}\frac{d^2\mathcal{M}_1}{d\phi^2} \right) \\ + \mathcal{M}_2 \left(\frac{a}{EI}\mathcal{M}_1 - \frac{1}{a}\frac{d\mathcal{V}_1}{d\phi} - \frac{1}{a}\frac{d^2\mathcal{W}_1}{d\phi^2} \right) \\ + \mathcal{N}_2 \left(\frac{a}{EA}\mathcal{N}_1 - \frac{d\mathcal{V}_1}{d\phi} + \mathcal{W}_1 \right) \Bigg\} d\phi \\ = -\omega_1^2 \int_0^{\phi^*} \rho a A \left(\mathcal{V}_1\mathcal{V}_2 + \mathcal{W}_1\mathcal{W}_2 \right) d\phi \end{aligned} \tag{62}$$

The left side of this equation may be expressed in operator notation, as follows:

$$\int_0^{\phi^*} \left(\sum_i \psi_i L_i \gamma_i \right) d\phi = -\omega_1^2 \int_0^{\phi^*} \rho a A \left(\mathcal{V}_1\mathcal{V}_2 + \mathcal{W}_1\mathcal{W}_2 \right) d\phi \tag{63}$$

Integrating the left side of Eq. (62) by parts, so that each term is integrated a number of times corresponding to the order of the ϕ derivative in the term, the following equation is obtained:

$$\begin{aligned} \int_0^{\phi^*} \left(\sum_i \gamma_i L_i \psi_i \right) d\phi \\ + \left[\frac{1}{a}\mathcal{M}_1 \left(\frac{d\mathcal{W}_2}{d\phi} + \mathcal{V}_2 \right) - \frac{1}{a}\mathcal{M}_2 \left(\frac{d\mathcal{W}_1}{d\phi} + \mathcal{V}_1 \right) \right. \\ \left. - \mathcal{V}_1\mathcal{N}_2 + \mathcal{V}_2\mathcal{N}_1 + \frac{1}{a}\mathcal{W}_1\frac{d\mathcal{M}_2}{d\phi} - \frac{1}{a}\mathcal{W}_2\frac{d\mathcal{M}_1}{d\phi} \right]_0^{\phi^*} \\ = -\omega_1^2 \int_0^{\phi^*} \rho a A \left(\mathcal{V}_1\mathcal{V}_2 + \mathcal{W}_1\mathcal{W}_2 \right) d\phi \end{aligned} \tag{64}$$

where the terms in the square brackets represent boundary conditions on the ring at $\phi = 0$ and $\phi = \phi^*$. When combined, Eq. (63) and (64) become

$$\int_0^{\phi^*} \left[\sum_i \left(\gamma_i L_i \psi_i - \psi_i L_i \gamma_i \right) \right] d\phi = -[\text{B.C.}]_0^{\phi^*} \tag{65}$$

If the boundary conditions on the right side of Eq. (65) can be satisfied so that the right side is identically zero, then Eq. (65) reduces to the symmetric form of Green's formula, and the differential operators L_i are self-adjoint, implying that the eigenvalues of the problem are real and distinct.

An equation may be derived similar to Eq. (63), except involving the integral

$$\int_0^{\phi^*} \left(\sum_i \gamma_i L_i \psi_i \right) d\phi$$

by reversing the subscripts in the original set of differential equations (58)–(61). The equation is

$$\int_0^{\phi^*} \left(\sum_i \gamma_i L_i \psi_i \right) d\phi = -\omega_2^2 \int_0^{\phi^*} \rho a A \left(\mathcal{V}_1\mathcal{V}_2 + \mathcal{W}_1\mathcal{W}_2 \right) d\phi \tag{66}$$

Combining Eq. (63), (65), and (66) results in the following equation:

$$\left(\omega_2^2 - \omega_1^2 \right) \int_0^{\phi^*} \rho a A \left(\mathcal{V}_1\mathcal{V}_2 + \mathcal{W}_1\mathcal{W}_2 \right) d\phi = [\text{B.C.}]_0^{\phi^*} \tag{67}$$

Again, if the boundary conditions can be chosen so that the right side of this equation is zero, then the equation reduces to an orthogonality condition for the modal functions $\mathcal{V}$ and $\mathcal{W}$. Certain combinations of terms in the expression for the boundary conditions [see Eq. (64)] will make the right sides of Eq. (65) and (67) zero. The possible boundary conditions at $\phi = 0, \phi^*$ are

$$\mathcal{M} = \mathcal{N} = \frac{d\mathcal{M}}{d\phi} = 0 \quad \text{(free end)} \tag{68}$$

$$\psi = \mathcal{N} = \frac{d\mathcal{M}}{d\phi} = 0 \quad \text{(clamped–free end)} \tag{69}$$

$$\mathcal{W} = \psi = \mathcal{N} = 0 \quad \text{(clamped–circumferentially guided end)} \tag{70}$$

$$\mathcal{V} = \psi = \frac{d\mathcal{M}}{d\phi} = 0 \quad \text{(clamped–radially guided end} \tag{71}$$

$$\mathcal{V} = \mathcal{W} = \mathcal{M} = 0 \quad \text{(pinned end)} \tag{72}$$

$$\mathcal{W} = \mathcal{N} = \mathcal{M} = 0 \quad \text{(pinned–circumferentially guided end)} \tag{73}$$

$$\mathcal{V} = \mathcal{M} = \frac{d\mathcal{M}}{d\phi} = 0 \quad \text{(pinned–radially guided end)} \tag{74}$$

$$\mathcal{V} = \mathcal{W} = \psi = 0 \quad \text{(clamped end)} \tag{75}$$

where ψ is the bending slope defined by

$$\psi = \frac{d\mathcal{W}}{d\phi} + \mathcal{V} \tag{76}$$

In Eq. (68)–(75), where hyphenated words are used to describe the ring constraints, the first word refers to the condition on rotational motion and the second word refers to the translational motion. Hence, "clamped–free" means the end of the ring cannot rotate but can translate radially and circumferentially.

If the ring has combinations of boundary conditions of the types listed above, then an orthogonality condition for the ring modal functions results:

$$\int_0^{\phi^*} \rho a A \left(\mathcal{V}_i \mathcal{V}_j + \mathcal{W}_i \mathcal{W}_j\right) d\phi = 0 \qquad (i \neq j) \tag{77}$$

This result is used in the following Section where forced-motion solutions of the ring equations, including extensional deformation, are derived.

C. Forced-Motion Solution

The coupled equations for forced motion are given by Eq. (10) and (11). In considering the homogeneous parts of these equations, it was found that the solution for $v(\phi, t)$ was of the form of Eq. (28) with a similar expression for $w(\phi, t)$. If these solutions are substituted into the coupled homogeneous equations, the resulting equations for the nth mode are

$$(1 + p)\frac{d^2\mathcal{V}_n}{d\phi^2} - \frac{d\mathcal{W}_n}{d\phi} + p\frac{d^3\mathcal{W}_n}{d\phi^3} = -\frac{\omega_n^2}{\omega_0^2}\mathcal{V}_n \tag{78}$$

$$\frac{d\mathcal{V}_n}{d\phi} - \mathcal{W}_n - p\frac{d^3\mathcal{V}_n}{d\phi^3} - p\frac{d^4\mathcal{W}_n}{d\phi^4} = -\frac{\omega_n^2}{\omega_0^2}\mathcal{W}_n \tag{79}$$

To determine particular solutions to Eq. (10) and (11), the space variation is assumed to be expressed by a summation of normal modes and the mode amplitudes by a normal coordinate $\eta_n(t)$. Then, solutions are

$$v(\phi, t) = \sum_{n=1}^{\infty} \mathcal{V}_n(\phi)\,\eta_n(t)$$

$$w(\phi, t) = \sum_{n=1}^{\infty} \mathcal{W}_n(\phi)\,\eta_n(t)$$

Substituting these solutions into Eq. (10) and (11) yields

$$\sum_{n=1}^{\infty}\left\{\left[(1 + p)\frac{d^2\mathcal{V}_n}{d\phi^2} - \frac{d\mathcal{W}_n}{d\phi} + p\frac{d^3\mathcal{W}_n}{d\phi^3}\right]\eta_n - \frac{1}{\omega_0^2}\mathcal{V}_n\ddot{\eta}_n\right\} = -\frac{a}{EA}F_T(\phi, t) \tag{80}$$

$$\sum_{n=1}^{\infty}\left\{\left[\frac{d\mathcal{V}_n}{d\phi} - \mathcal{W}_n - p\frac{d^3\mathcal{V}_n}{d\phi^3} - p\frac{d^4\mathcal{W}_n}{d\phi^4}\right]\eta_n - \frac{1}{\omega_0^2}\mathcal{W}_n\ddot{\eta}_n\right\} = -\frac{a}{EA}F_R(\phi, t) \tag{81}$$

where each dot represents one derivative with respect to time. Equations (78) and (79) can be used to reduce Eq. (80) and (81) since it is assumed that when the ring is driven by an external force, its displacement is defined by a superposition of normal modes. Hence

$$\sum_{n=1}^{\infty}\left[\mathcal{V}_n\frac{1}{\omega_0^2}\ddot{\eta}_n + \frac{\omega_n^2}{\omega_0^2}\mathcal{V}_n\eta_n\right] = \frac{a}{EA}F_T(\phi, t) \tag{82}$$

$$\sum_{n=1}^{\infty}\left[\mathcal{W}_n\frac{1}{\omega_0^2}\ddot{\eta}_n + \frac{\omega_n^2}{\omega_0^2}\mathcal{W}_n\eta_n\right] = \frac{a}{EA}F_R(\phi, t) \tag{83}$$

Premultiplying Eq. (82) and (83) by the factors

$$\mathcal{V}_m d\phi$$

and

$$\mathcal{W}_m d\phi$$

respectively, and integrating over the arc length of the ring, results in

$$\sum_{n=1}^{\infty}\left[\int_0^{\phi^*} \mathcal{V}_n\mathcal{V}_m d\phi \{\ddot{\eta}_n + \omega_n^2\eta_n\}\right] = \frac{a\omega_0^2}{EA}\int_0^{\phi^*} F_T(\phi, t)\,\mathcal{V}_m d\phi$$

$$\sum_{n=1}^{\infty}\left[\int_0^{\phi^*} \mathcal{W}_n\mathcal{W}_m d\phi \{\ddot{\eta}_n + \omega_n^2\eta_n\}\right] = \frac{a\omega_0^2}{EA}\int_0^{\phi^*} F_R(\phi, t)\,\mathcal{W}_m d\phi$$

Adding these equations and applying the orthogonality condition derived in Section IV-B results in the following:

$$\ddot{\eta}_n + \omega_n^2\eta_n = Q_n(t) \tag{84}$$

where

$$Q_n(t) = \frac{\int_0^{\phi^*} [F_T(\phi, t)\,\mathcal{V}_n + F_R(\phi, t)\,\mathcal{W}_n]\,d\phi}{\rho a A\int_0^{\phi^*} (\mathcal{V}_n^2 + \mathcal{W}_n^2)\,d\phi} \tag{85}$$

The solution to Eq. (84) may be written in integral form as follows:

$$\eta_n(t) = \frac{1}{\omega_n}\int_0^t Q_n(\tau)\sin\omega_n(t-\tau)\,d\tau + C_{n_1}\sin\omega_n t + C_{n_2}\cos\omega_n t$$

Hence, the solutions for the displacements are

$$v(\phi, t) = \sum_{n=1}^{\infty} \mathcal{V}_n(\phi) \tag{86}$$

$$\times\left[\frac{1}{\omega_n}\int_0^t Q_n(\tau)\sin\omega_n(t-\tau)\,d\tau + C_{n_1}\sin\omega_n t + C_{n_2}\cos\omega_n t\right]$$

$$w(\phi, t) = \sum_{n=1}^{\infty} \mathcal{W}_n(\phi) \tag{87}$$

$$\times\left[\frac{1}{\omega_n}\int_0^t Q_n(\tau)\sin\omega_n(t-\tau)\,d\tau + C_{n_1}\sin\omega_n t + C_{n_2}\cos\omega_n t\right]$$

The coefficients C_{n_i} are non-zero only if the initial displacements and velocities are non-zero. Assuming the following set of initial conditions:

$$\begin{aligned} v(\phi, 0) &= v_0 \\ \dot{v}(\phi, 0) &= \dot{v}_0 \\ w(\phi, 0) &= w_0 \\ \dot{w}(\phi, 0) &= \dot{w}_0 \end{aligned} \tag{88}$$

then, for the initial conditions on $v(\phi, t)$, Eq. (86)[2] reduces to

$$\int_0^{\phi^*} v_0\mathcal{V}_m d\phi = \sum_{n=1}^{\infty} C_{n_2}\int_0^{\phi^*} \mathcal{V}_n\mathcal{V}_m d\phi \tag{89}$$

$$\int_0^{\phi^*} \dot{v}_0\mathcal{V}_m d\phi = \sum_{n=1}^{\infty} \omega_n C_{n_1}\int_0^{\phi^*} \mathcal{V}_n\mathcal{V}_m d\phi \tag{90}$$

and for the initial conditions on $w(\phi, t)$, Eq. (87) reduces to

$$\int_0^{\phi^*} w_0\mathcal{W}_m d\phi = \sum_{n=1}^{\infty} C_{n_2}\int_0^{\phi^*} \mathcal{W}_n\mathcal{W}_m d\phi \tag{91}$$

$$\int_0^{\phi^*} \dot{w}_0\mathcal{W}_m d\phi = \sum_{n=1}^{\infty} \omega_n C_{n_1}\int_0^{\phi^*} \mathcal{W}_n\mathcal{W}_m d\phi \tag{92}$$

Adding Eq. (90) and (92) and noting orthogonality, the solution for C_{n_1} is

$$C_{n_1} = \frac{\int_0^{\phi^*} (\dot{v}_0\mathcal{V}_n + \dot{w}_0\mathcal{W}_n)\,d\phi}{\omega_n\int_0^{\phi^*} (\mathcal{W}_n^2 + \mathcal{V}_n^2)\,d\phi} \tag{93}$$

and similarly for C_{n_2}, the solution is

$$C_{n_2} = \frac{\int_0^{\phi^*} (v_0\mathcal{V}_n + w_0\mathcal{W}_n)\,d\phi}{\int_0^{\phi^*} (\mathcal{W}_n^2 + \mathcal{V}_n^2)\,d\phi} \tag{94}$$

If these expressions for the constants are substituted into Eq. (86) and (87), the complete solution is obtained.

In a particular problem, if the applied force is representable in Eq. (85), then the displacement solutions may be obtained by evaluating the various integrals appearing in the solution equations.

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[2]The equation was multiplied by the factor $\mathcal{V}_m d\phi$ and integrated over the arc length of the ring.

V. THE INEXTENSIONAL EQUATIONS

A. Derivation of Equations of Motion

As already discussed, the restriction that extension of the neutral axis of the ring be small compared to bending deformation appears to be a valid assumption for ring motion in the lower modes. This approximation leads to a relation between the radial displacement and the tangential slope, namely

$$w = \frac{\partial v}{\partial \phi} \tag{95}$$

which allows uncoupling of the differential equations of motion. To derive the uncoupled equations, consider first the equilibrium Eq. (4) and (5). From Eq. (5)

$$N = -\frac{1}{a}\frac{\partial^2 M}{\partial \phi^2} + \rho a A \frac{\partial^2 w}{\partial t^2} - F_R(\phi, t) \tag{96}$$

Substituting this result into Eq. (4) yields

$$-\frac{1}{a}\frac{\partial^3 M}{\partial \phi^3} - \frac{1}{a}\frac{\partial M}{\partial \phi} - \rho a A \frac{\partial^2 v}{\partial t^2} + \rho a A \frac{\partial^3 w}{\partial \phi \partial t^2}$$

$$= -F_T(\phi, t) + \frac{\partial}{\partial \phi}[F_R(\phi, t)] \tag{97}$$

But,

$$M = -\frac{EI}{a^2}\left(\frac{\partial v}{\partial \phi} + \frac{\partial^2 w}{\partial \phi^2}\right) \tag{98}$$

Therefore, substituting Eq. (95) and (98) into (97) results in

$$\frac{\partial^6 v}{\partial \phi^6} + 2\frac{\partial^4 v}{\partial \phi^4} + \frac{\partial^2 v}{\partial \phi^2} + \frac{1}{p\omega_0^2}\frac{\partial^4 v}{\partial \phi^2 \partial t^2} - \frac{1}{p\omega_0^2}\frac{\partial^2 v}{\partial t^2}$$

$$= \frac{a^3}{EI}\left\{\frac{\partial}{\partial \phi}[F_R(\phi, t)] - F_T(\phi, t)\right\} \tag{99}$$

Similarly, the equation in terms of the dependent variable w is

$$\frac{\partial^6 w}{\partial \phi^6} + 2\frac{\partial^4 w}{\partial \phi^4} + \frac{\partial^2 w}{\partial \phi^2} + \frac{1}{p\omega_0^2}\frac{\partial^4 w}{\partial \phi^2 \partial t^2} - \frac{1}{p\omega_0^2}\frac{\partial^2 w}{\partial t^2}$$

$$= \frac{a^3}{EI}\left\{\frac{\partial^2}{\partial \phi^2}[F_R(\phi, t)] - \frac{\partial}{\partial \phi}[F_T(\phi, t)]\right\} \tag{100}$$

B. Solution of the Homogeneous Equations

Solutions of the homogeneous parts of Eq. (99) and (100) have been derived by several authors (Ref. 2, 4, 5) so that detailed derivations need not be repeated. It will suffice to mention only that the solutions parallel those carried out for the extensional case. The results are

Case I. $\xi_n < \xi_L^*$:

The modal functions are of the form

$$\mathcal{V}_n = A_{n_1}\cos d_n\phi + A_{n_2}\sin d_n\phi + A_{n_3}\cos e_n\phi + A_{n_4}\sin e_n\phi + A_{n_5}\cos f_n\phi + A_{n_6}\sin f_n\phi \tag{101}$$

$$\mathcal{W}_n = -A_{n_1}d_n\sin d_n\phi + A_{n_2}d_n\cos d_n\phi - A_{n_3}e_n\sin e_n\phi + A_{n_4}e_n\cos e_n\phi - A_{n_5}f_n\sin f_n\phi + A_{n_6}f_n\cos f_n\phi \tag{102}$$

Case II. $\xi_L^* < \xi_n < \xi_U^*$:

$$\mathcal{V}_n = A_{n_1}\cos d_n\phi + A_{n_2}\sin d_n\phi + A_{n_3}\cos \mu_n\phi\cosh \delta_n\phi + A_{n_4}\sin \mu_n\phi\cosh \delta_n\phi + A_{n_5}\cos \mu_n\phi\sinh \delta_n\phi + A_{n_6}\sin \mu_n\phi\sinh \delta_n\phi \tag{103}$$

$$\mathcal{W}_n = -d_nA_{n_1}\sin d_n\phi + d_nA_{n_2}\cos d_n\phi + A_{n_3}(-\mu_n\sin \mu_n\phi\cosh \delta_n\phi + \delta_n\cos \mu_n\phi\sinh \delta_n\phi) + A_{n_4}(\mu_n\cos \mu_n\phi\cosh \delta_n\phi + \delta_n\sin \mu_n\phi\sinh \delta_n\phi) + A_{n_5}(-\mu_n\sin \mu_n\phi\sinh \delta_n\phi + \delta_n\cos \mu_n\phi\cosh \delta_n\phi) + A_{n_6}(\mu_n\cos \mu_n\phi\sinh \delta_n\phi + \delta_n\sin \mu_n\phi\cosh \delta_n\phi) \tag{104}$$

Case III. $\xi_n > \xi_U^*$

$$\mathcal{V}_n = A_{n_1}\cos d_n\phi + A_{n_2}\sin d_n\phi + A_{n_3}\cosh e_n\phi + A_{n_4}\sinh e_n\phi + A_{n_5}\cosh f_n\phi + A_{n_6}\sinh f_n\phi \tag{105}$$

$$\mathcal{W}_n = -d_nA_{n_1}\sin d_n\phi + d_nA_{n_2}\cos d_n\phi + e_nA_{n_3}\sinh e_n\phi + e_nA_{n_4}\cosh e_n\phi + f_nA_{n_5}\sinh f_n\phi + f_nA_{n_6}\cosh f_n\phi \tag{106}$$

Depending upon the value of ξ_n, the various constants $d_n, \delta_n, \mu_n, c_n$, and f_n are related to the roots of the following sixth-order equation:

$$m_n^6 + 2\,m_n^4 + (1 - \xi_n)\,m_n^2 + \xi_n = 0 \qquad (107)$$

These relationships were derived for the extensional case in Section IV-A, and apply equally well in the present case. The only difference is that in Eq. (107), the parameter p does not appear as in the extensional equation [Eq. (32)] so that there are only two change-over values of ξ_n, namely $\xi_L^* = 0.1134$ and $\xi_M^* = 17.637$, independent of the ring geometry.

The procedure for computing numerically the eigenvalues and the constants in the modal functions for particular types of constrained rings is the same as outlined in Section IV-A, and a specific example using the inextensional equations is outlined in the Appendix.

C. Orthogonality of the Ring Modes

The orthogonality of the inextensional ring modes can be demonstrated using the same techniques that were employed in establishing orthogonality of the extensional modes (Section IV-B). However, since $\mathcal{N}$ and its derivatives with respect to ϕ are assumed equal to zero throughout the ring, it is advantageous to establish the orthogonality relation by operating on one of the sixth-order differential equations in $\mathcal{V}$ or $\mathcal{W}$, rather than the coupled equations having variables $\mathcal{M}$, $d\mathcal{M}/d\phi$, $\mathcal{V}$, and $\mathcal{W}$. Therefore, assume a solution

$$v = \mathcal{V}_1(\phi)\, e^{i\omega_1 t} \qquad (108)$$

Substitute this solution into the homogeneous part of Eq. (99)

$$\frac{d^6\mathcal{V}_1}{d\phi^6} + 2\frac{d^4\mathcal{V}_1}{d\phi^4} + (1 - \xi_1)\frac{d^2\mathcal{V}_1}{d\phi^2} + \xi_1\mathcal{V}_1 = 0 \qquad (109)$$

where

$$\xi_1 = \frac{\rho a^4 A}{EI}\,\omega_1^2 \qquad (110)$$

Multiplying Eq. (109) by a second solution $\mathcal{V}_2$ and integrating over the arc length of the ring results in

$$\int_0^{\phi^*} \mathcal{V}_2\left[\frac{d^6\mathcal{V}_1}{d\phi^6} + 2\frac{d^4\mathcal{V}_1}{d\phi^4} + (1 - \xi_1)\frac{d^2\mathcal{V}_1}{d\phi^2} + \xi_1\mathcal{V}_1\right] d\phi = 0 \qquad (111)$$

Integrating each product of the integrand by parts a number of times corresponding to the power of the ϕ derivative, yields

$$\left[\mathcal{V}_2\frac{d^5\mathcal{V}_1}{d\phi^5} - \frac{d\mathcal{V}_2}{d\phi}\frac{d^4\mathcal{V}_1}{d\phi^4} + \frac{d^2\mathcal{V}_2}{d\phi^2}\frac{d^3\mathcal{V}_1}{d\phi^3} + 2\mathcal{V}_2\frac{d^3\mathcal{V}_1}{d\phi^3} - 2\frac{d\mathcal{V}_2}{d\phi}\frac{d^2\mathcal{V}_1}{d\phi^2} + (1 - \xi_1)\mathcal{V}_2\frac{d\mathcal{V}_1}{d\phi}\right]_0^{\phi^*}$$
$$- \int_0^{\phi^*}\left[\frac{d^3\mathcal{V}_1}{d\phi^3}\frac{d^3\mathcal{V}_2}{d\phi^3} - 2\frac{d^2\mathcal{V}_1}{d\phi^2}\frac{d^2\mathcal{V}_2}{d\phi^2} + \frac{d\mathcal{V}_1}{d\phi}\frac{d\mathcal{V}_2}{d\phi}\right] d\phi$$
$$= -\xi_1\int_0^{\phi^*}\left(\frac{d\mathcal{V}_1}{d\phi}\frac{d\mathcal{V}_2}{d\phi} + \mathcal{V}_1\mathcal{V}_2\right) d\phi \qquad (112)$$

where the assumption is made that the ring is uniform so that ξ_1 can be factored from the integrands. A second equation may be defined by reversing the subscripts in Eq. (112). If this equation is subtracted from Eq. (112) and if the terms representing the boundary conditions are grouped according to the definitions for $\mathcal{H}$, $d\mathcal{H}/d\phi$, and $\mathcal{N}$, namely

$$\mathcal{H} = \frac{EI}{a^2}\left(\frac{d\mathcal{V}}{d\phi} + \frac{d^3\mathcal{V}}{d\phi^3}\right) \qquad (113)$$

$$\frac{d\mathcal{H}}{d\phi} = \frac{EI}{a^2}\left(\frac{d^2\mathcal{V}}{d\phi^2} + \frac{d^4\mathcal{V}}{d\phi^4}\right) \qquad (114)$$

$$\frac{a^3}{EI}\mathcal{N} = -\xi\frac{d\mathcal{V}}{d\phi} + \frac{d^3\mathcal{V}}{d\phi^3} + \frac{d^5\mathcal{V}}{d\phi^5} \qquad \text{[from Eq. (59)]} \qquad (115)$$

the following equation is obtained:

$$\left[(\mathcal{V}_2\mathcal{N}_1 - \mathcal{V}_1\mathcal{N}_2) - \frac{1}{a}\mathcal{M}_2\left(\mathcal{V}_1 + \frac{d^2\mathcal{V}_1}{d\phi^2}\right)\right.$$
$$\left. + \frac{1}{a}\mathcal{M}_1\left(\mathcal{V}_2 + \frac{d^2\mathcal{V}_2}{d\phi^2}\right) - \frac{1}{a}\frac{d\mathcal{V}_2}{d\phi}\frac{d\mathcal{M}_1}{d\phi} + \frac{1}{a}\frac{d\mathcal{V}_1}{d\phi}\frac{d\mathcal{M}_2}{d\phi}\right]_0^{\phi^*}$$
$$= (\omega_2^2 - \omega_1^2)\int_0^{\phi^*}\rho a A\left(\frac{d\mathcal{V}_1}{d\phi}\frac{d\mathcal{V}_2}{d\phi} + \mathcal{V}_1\mathcal{V}_2\right) d\phi \qquad (116)$$

Using Eq. (95), the orthogonality equation becomes

$$\int_0^{\phi^*}\rho a A\,(\mathcal{V}_i\mathcal{V}_j + \mathcal{W}_i\mathcal{W}_j)\,d\phi = 0 \qquad (i \neq j) \qquad (117)$$

In defining the possible types of homogeneous boundary conditions, the following relations are used:

$$\mathcal{W} = \frac{d\mathcal{V}}{d\phi}$$

$$\psi = \mathcal{V} + \frac{d\mathcal{W}}{d\phi} = \mathcal{V} + \frac{d^2\mathcal{V}}{d\phi^2}$$

Furthermore, since $\mathcal{N}$ is assumed to be zero throughout the ring, $d\mathcal{N}/d\phi$ is also zero throughout, so that from Eq. (58) the following relation is obtained:

$$\frac{d\mathcal{H}}{d\phi} = \rho a^2 A \omega^2 Q'$$

Using these relations and by examining the terms in the square brackets in Eq. (116), the following expressions for the boundary conditions are obtained:

1. Free end:

$$Q' = 0 \qquad \left(\frac{d\mathcal{H}}{d\phi} = 0\right)$$

$$\frac{dQ'}{d\phi} + \frac{d^3Q'}{d\phi^3} = 0 \qquad (\mathcal{M} = 0) \tag{118}$$

$$(1 + \xi)\frac{dQ'}{d\phi} - \frac{d^5Q'}{d\phi^5} = 0 \qquad (\mathcal{N} = 0, \text{ from Eq. (59)})$$

2. Clamped–free end:

$$Q' = 0 \qquad \left(\frac{d\mathcal{H}}{d\phi} = 0\right)$$

$$\frac{d^2Q'}{d\phi^2} = 0 \qquad (\psi = 0) \tag{119}$$

$$-\xi\frac{dQ'}{d\phi} + \frac{d^3Q'}{d\phi^3} + \frac{d^5Q'}{d\phi^5} = 0 \qquad (\mathcal{N} = 0, \text{ from Eq. (59)})$$

3. Clamped–circumferentially guided end:

$$\frac{dQ'}{d\phi} = 0 \qquad (\mathcal{W} = 0)$$

$$Q' + \frac{d^2Q'}{d\phi^2} = 0 \qquad (\psi = 0) \tag{120}$$

$$\frac{d^3Q'}{d\phi^3} + \frac{d^5Q'}{d\phi^5} = 0 \qquad (\mathcal{N} = 0)$$

4. Clamped–radially guided end:

$$Q' = 0$$

$$\frac{d^2Q'}{d\phi^2} = 0 \qquad (\psi = 0) \tag{121}$$

$$\frac{d^4Q'}{d\phi^4} = 0 \qquad \left(\frac{d\mathcal{M}}{d\phi} = 0\right)$$

5. Pinned end:

$$Q' = 0$$

$$\frac{dQ'}{d\phi} = 0 \qquad (\mathcal{W} = 0) \tag{122}$$

$$\frac{d^3Q'}{d\phi^3} = 0 \qquad (\mathcal{H} = 0)$$

6. Pinned–circumferentially guided end:

$$\frac{dQ'}{d\phi} = 0 \qquad (\mathcal{W} = 0)$$

$$\frac{d^3Q'}{d\phi^3} = 0 \qquad (\mathcal{H} = 0) \tag{123}$$

$$\frac{d^5Q'}{d\phi^5} = 0 \qquad (\mathcal{N} = 0)$$

7. Pinned–radially guided end:

$$Q' = 0$$

$$\frac{dQ'}{d\phi} + \frac{d^3Q'}{d\phi^3} = 0 \qquad (\mathcal{H} = 0) \tag{124}$$

$$\frac{d^2Q'}{d\phi^2} + \frac{d^4Q'}{d\phi^4} = 0 \qquad \left(\frac{d\mathcal{H}}{d\phi} = 0\right)$$

8. Clamped end:

$$Q' = 0$$

$$\frac{dQ'}{d\phi} = 0 \qquad (\mathcal{W} = 0) \tag{125}$$

$$\frac{d^2Q'}{d\phi^2} = 0 \qquad (\psi = 0)$$

D. Forced-Motion Solution

General solutions to Eq. (99) and (100) may be derived separately; however, an actual derivation need be performed only for the dependent variable v, since corresponding solutions for w may be determined from Eq. (95).

Assume the forced motion can be represented by a summation of ring normal modes, these modes being derived from the equation (v variable)

$$\frac{d^6Q'_n}{d\phi^6} + 2\frac{d^4Q'_n}{d\phi^4} + \frac{d^2Q'_n}{d\phi^2} = \xi_n\frac{d^2Q'_n}{d\phi^2} - \xi_n Q'_n \tag{126}$$

The particular solution to Eq. (99) is then

$$v(\phi, t) = \sum_{n=1}^{\infty} Q'_n(\phi)\,\eta_n(t) \tag{127}$$

Substituting Eq. (127), then Eq. (126), into Eq. (99) results in

$$\sum_{n=1}^{\infty} \rho a A \left(\frac{d^2 \mathcal{V}_n}{d\phi^2} - \mathcal{V}_n \right) (\ddot{\eta}_n + \omega_n^2 \eta_n) = \frac{\partial}{\partial \phi} F_R(\phi, t) - F_T(\phi, t)$$

To simplify this equation, it is necessary to multiply the equation by the factor $\mathcal{V}_m d\phi$, integrate over the ring arc length, then integrate the integrands by parts once, and finally, to apply the condition expressed by Eq. (95). The resulting equation is

$$\sum_{n=1}^{\infty} \left\{ \rho a A \left[\mathcal{V}_m \frac{d\mathcal{V}_n}{d\phi} \Big|_0^{\phi^*} - \int_0^{\phi^*} (\mathcal{W}_n \mathcal{W}_m + \mathcal{V}_n \mathcal{V}_m)\, d\phi \right] \times (\ddot{\eta}_n + \omega_n^2 \eta_n) \right\} = \int_0^{\phi^*} \mathcal{V}_m \left\{ \frac{\partial}{\partial \phi} F_R(\phi, t) - F_T(\phi, t) \right\} d\phi \qquad (128)$$

For all boundary conditions the term

$$\mathcal{V}_m \frac{d\mathcal{V}_n}{d\phi} \Big|_0^{\phi^*}$$

vanishes, since for each type of boundary condition, either $\mathcal{V}$ or $d\mathcal{V}/d\phi$ is zero. With this information, Eq. (128) can be simplified to the following

$$\ddot{\eta}_n + \omega_n^2 \eta_n = Q_n(t) \qquad (129)$$

where

$$Q_n(t) = -\frac{\int_0^{\phi^*} \mathcal{V}_n \left\{ \frac{\partial}{\partial \phi} F_R(\phi, t) - F_T(\phi, t) \right\} d\phi}{\rho a A \int_0^{\phi^*} (\mathcal{W}_n^2 + \mathcal{V}_n^2)\, d\phi} \qquad (130)$$

The form of Eq. (129) is the same as that of Eq. (84); thus, the procedure outlined for solving Eq. (84) applies equally well in solving this equation.

It is interesting to note that in Eq. (130), the radial force must be differentiable in ϕ, which is a necessary condition for not exciting extensional modes.

VI. DISCUSSION

The solutions derived for the extensional and inextensional vibration of a circular ring have not been dependent upon the numerical value of ϕ^*, which is the upper limit on the various integrals and represents the angular location of one end of an incomplete ring, the other end being located at position $\phi = 0$. If $\phi^* = \pi$ then the eigenvalues and modal functions for a half-ring will be obtained; however, for certain sets of boundary conditions, these results are applicable to complete rings since, in the vibration of complete rings, symmetry can be established about a diametrical axis such as $\phi = 0, \pi$.

To cite an example, the mode shapes and natural frequencies of a complete ring clamped at one point on its circumference may be determined by solving for the symmetric and asymmetric cases of a half-ring. The corresponding sets of boundary conditions for the half-ring are those of Eq. (75) and (71) for the symmetric modes and those of Eq. (75) and (73) for the asymmetric modes.

Based upon the derivation of the modal functions and boundary conditions in this analysis, if a ring (complete or incomplete) is constrained at more than two points on its circumference, the problem is essentially the same as that of a beam on multiple supports. For this type of problem, it is necessary to define modal functions for each span of ring between supports (constraints), and to match boundary conditions at these intermediate supports, as well as at the ends of the ring segment or at angles of symmetry.

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APPENDIX

Symmetric Mode Shapes and Natural Frequencies of a Complete Circular Ring Clamped at One Point on Its Circumference

A. Computed Results

The inextensional theory is applied to the case of a complete circular ring clamped at one point on its circumference ("point-clamped" ring). The clamped condition is assumed to be at the angular position $\phi = 0$. Then, since the symmetric mode shapes are to be determined, the second boundary constraint is the clamped–radially guided end at $\phi^* = \pi$. Hence, the boundary conditions are

$$\left.\begin{aligned} \mathcal{V}_n &= 0 \\ \frac{d\mathcal{V}_n}{d\phi} &= 0 \\ \frac{d^2\mathcal{V}_n}{d\phi^2} &= 0 \end{aligned}\right\} \quad \text{at } \phi = 0 \tag{A-1}$$

$$\left.\begin{aligned} \mathcal{V}_n &= 0 \\ \frac{d^2\mathcal{V}_n}{d\phi^2} &= 0 \\ \frac{d^4\mathcal{V}_n}{d\phi^4} &= 0 \end{aligned}\right\} \quad \text{at } \phi^* = \pi \tag{A-2}$$

The generalized modal functions for $\mathcal{V}_n(\phi)$ are given by Eq. (103) and (105) only, since there are no roots in the range $\xi_n < \xi_L^*$. Using these expressions with the boundary conditions yields the following frequency matrices:

1. $\xi_L^* < \xi_n < \xi_U^*$

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 \\
0 & d_n & 0 & \mu_n & 0 & \delta_n \\
-d_n^2 & 0 & \delta_n^2 - \mu_n^2 & 0 & 2\delta_n\mu_n & 0 \\
\cos d_n\pi & \sin d_n\pi & \cos \mu_n\pi \cosh \delta_n\pi & \sin \mu_n\pi \cosh \delta_n\pi & \cos \mu_n\pi \sinh \delta_n\pi & \sin \mu_n\pi \sinh \delta_n\pi \\
-d_n^2 \cos d_n\pi & -d_n^2 \sin d_n\pi & \begin{array}{l}(\delta_n^2 - \mu_n^2) \cos \mu_n\pi \cosh \delta_n\pi \\ - 2\delta_n\mu_n \sin \mu_n\pi \sinh \delta_n\pi\end{array} & \begin{array}{l}(\delta_n^2 - \mu_n^2) \sin \mu_n\pi \cosh \delta_n\pi \\ + 2\delta_n\mu_n \cos \mu_n\pi \sinh \delta_n\pi\end{array} & \begin{array}{l}(\delta_n^2 - \mu_n^2) \cos \mu_n\pi \sinh \delta_n\pi \\ + 2\delta_n\mu_n \sin \mu_n\pi \cosh \delta_n\pi\end{array} & \begin{array}{l}(\delta_n^2 - \mu_n^2) \sin \mu_n\pi \sinh \delta_n\pi \\ + 2\delta_n\mu_n \cos \mu_n\pi \cosh \delta_n\pi\end{array} \\
d_n^4 \cos d_n\pi & d_n^4 \sin d_n\pi & \begin{array}{l}(\mu_n^4 - 6\delta_n^2\mu_n^2 + \delta_n^4) \\ \cdot \cos \mu_n\pi \cosh \delta_n\pi \\ + 4\mu_n\delta_n (\mu_n^2 - \delta_n^2) \\ \cdot \sin \mu_n\pi \sinh \delta_n\pi\end{array} & \begin{array}{l}(\mu_n^4 - 6\delta_n^2\mu_n^2 + \delta_n^4) \\ \cdot \sin \mu_n\pi \cosh \delta_n\pi \\ - 4\mu_n\delta_n (\mu_n^2 - \delta_n^2) \\ \cdot \cos \mu_n\pi \sinh \delta_n\pi\end{array} & \begin{array}{l}(\mu_n^4 - 6\delta_n^2\mu_n^2 + \delta_n^4) \\ \cdot \cos \mu_n\pi \sinh \delta_n\pi \\ + 4\mu_n\delta_n (\mu_n^2 - \delta_n^2) \\ \cdot \sin \mu_n\pi \cosh \delta_n\pi\end{array} & \begin{array}{l}(\mu_n^4 - 6\delta_n^2\mu_n^2 + \delta_n^4) \\ \cdot \sin \mu_n\pi \sinh \delta_n\pi \\ - 4\mu_n\delta_n (\mu_n^2 - \delta_n^2) \\ \cdot \cos \mu_n\pi \cosh \delta_n\pi\end{array}
\end{bmatrix} \tag{A-3}
$$

2. $\xi_n > \xi_U^*$

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & d_n & 0 & e_n & 0 & f_n \\
-d_n^2 & 0 & e_n^2 & 0 & f_n^2 & 0 \\
\cos d_n\pi & \sin d_n\pi & \cosh e_n\pi & \sinh e_n\pi & \cosh f_n\pi & \sinh f_n\pi \\
-d_n^2 \cos d_n\pi & -d_n^2 \sin d_n\pi & e_n^2 \cosh e_n\pi & e_n^2 \sinh e_n\pi & f_n^2 \cosh f_n\pi & f_n^2 \sinh f_n\pi \\
d_n^4 \cos d_n\pi & d_n^4 \sin d_n\pi & e_n^4 \cosh e_n\pi & e_n^4 \sinh e_n\pi & f_n^4 \cosh f_n\pi & f_n^4 \sinh f_n\pi
\end{bmatrix} \tag{A-4}
$$

where the parameters d_n, δ_n, μ_n, e_n, and f_n are related to the roots of the equation

$$m_n^6 + 2\,m_n^4 + (1 - \xi_n)\,m_n^2 + \xi_n = 0 \qquad \text{(A-5)}$$

as specified in Section IV-A. Carrying out the procedure outlined in Section IV-A for determining the eigenvalues, the following results were obtained:

$\xi_1 = 2.544$	$d_1 = 1.689$	$\delta_1 = 0.828$	$\mu_1 = 0.509$
$\xi_2 = 33.10$	$d_2 = 2.673$	$e_2 = 1.995$	$f_2 = 1.079$
$\xi_3 = 146.9$	$d_3 = 3.682$	$e_3 = 3.245$	$f_3 = 1.014$

The modal functions were normalized with respect to the integral condition that

$$\int_0^{\pi} \left[\mathcal{V}_n^2 + \left(\frac{d\mathcal{V}_n}{d\phi} \right)^2 \right] d\phi = \pi$$

The resulting expressions for the $\mathcal{V}_n$ are

$$\begin{aligned}
\mathcal{V}_1 = {} & 0.572 \cos 1.69\phi + 0.387 \sin 1.69\phi \\
& - 0.572 \cos 0.509\phi \cosh 0.828\phi \\
& - 2.21 \sin 0.509\phi \cosh 0.828\phi \\
& + 0.565 \cos 0.509\phi \sinh 0.828\phi \\
& + 2.23 \sin 0.509\phi \sinh 0.828\phi \\
\mathcal{V}_2 = {} & 0.426 \cos 2.67\phi + 0.257 \sin 2.67\phi \\
& + 1.26 \cosh 2.00\phi - 1.26 \sinh 2.00\phi \\
& - 1.68 \cosh 1.08\phi + 1.69 \sinh 1.08\phi \\
\mathcal{V}_3 = {} & 0.312 \cos 3.68\phi + 0.204 \sin 3.68\phi \\
& + 0.480 \cosh 3.25\phi - 0.480 \sinh 3.25\phi \\
& - 0.792 \cosh 1.01\phi + 0.795 \sinh 1.01\phi
\end{aligned} \qquad \text{(A-6)}$$

B. Experiment

To verify the theoretical results, a ring was tested on an electromechanical vibration exciter. The ring was made from hot-rolled mild-steel stock and had the following dimensions: width, 1.0 in.; thickness, 0.25 in.; diameter, 20.0 in. The clamped constraint was made by using contoured washers between the ring and armature of the exciter (see Fig. A-1). A photograph of the test setup is shown in Fig. A-2. Input and output information monitored during the test was measured with piezo-electric accelerometers. One accelerometer was mounted on the armature of the exciter to measure the input acceleration. A second accelerometer was mounted on a small block which was moved along the circumference of the ring to make measurements of the radial acceleration. Resonances were detected by monitoring the signal from the output accelerometer on a vacuum-tube voltmeter as frequency sweeps were conducted. When the exciter was set at a resonance, the frequency was measured on an electronic counter that was connected to the amplifier circuit of the exciter system.

Experimental and calculated natural frequencies and the percentage differences for the three lowest flexural modes are presented in Table A-1.

Table A-1. Comparison of calculated and experimental values of natural frequencies for the "point-clamped" ring

Mode number, n	Calculated frequency, cps	Experimental frequency, cps	Difference, %
1	39	40	2.5
2	140	140	0
3	296	296	0

The three lowest mode shapes were computed from Eq. (A-6) and the plotted results are shown in Fig. A-3, A-4, and A-5. Experimental points are also shown in these Figures, and it may be concluded from these results that excellent agreement between theory and experiment was obtained.

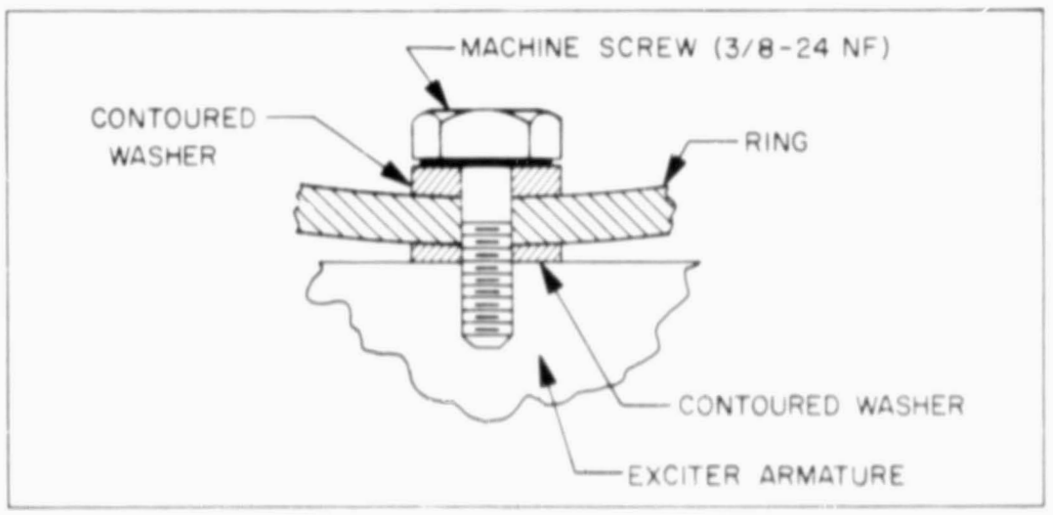

Fig. A-1. Detail of clamped constraint for the "point-clamped" ring

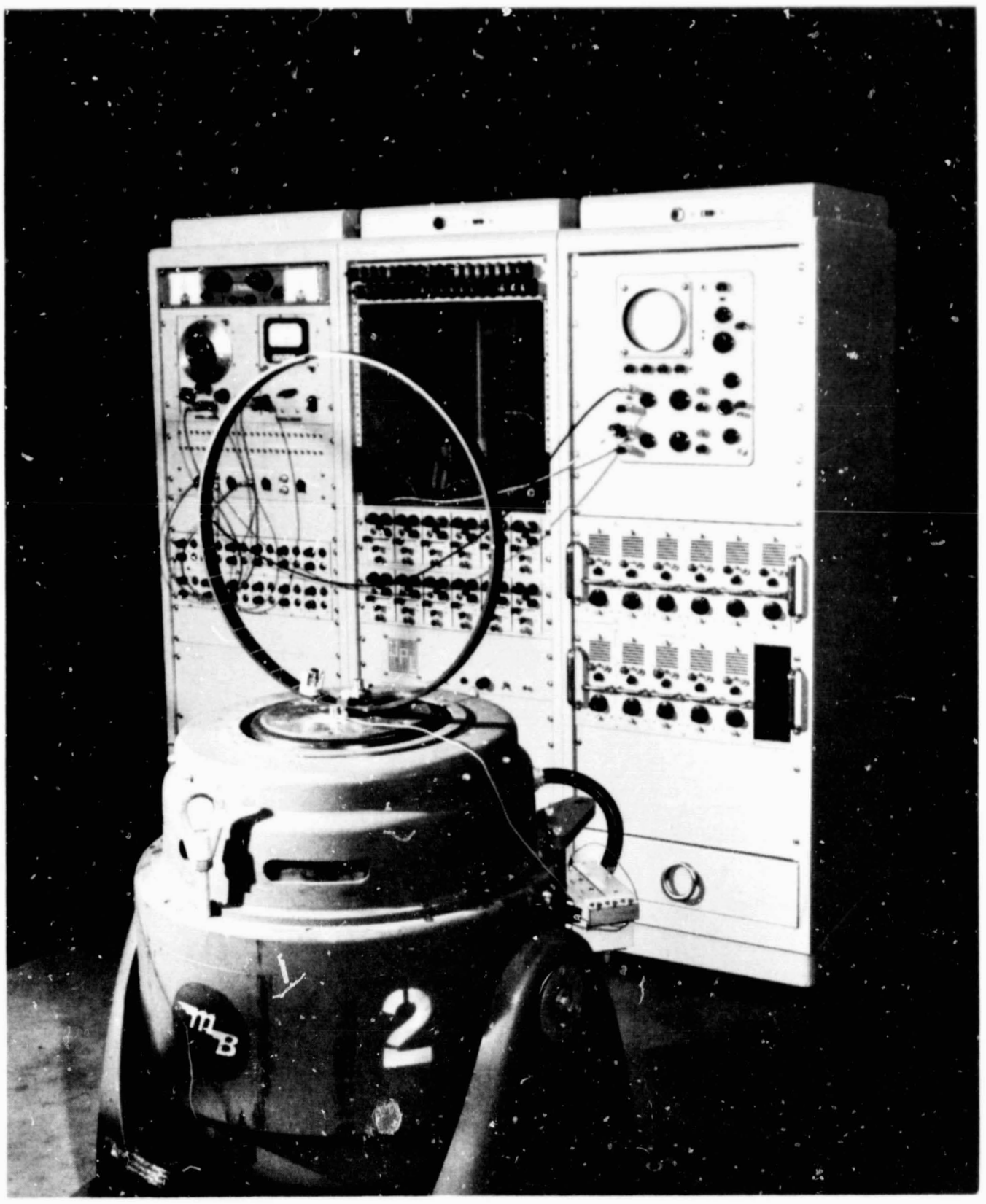

Fig. A-2. Test setup for the vibration test of the "point-clamped" ring

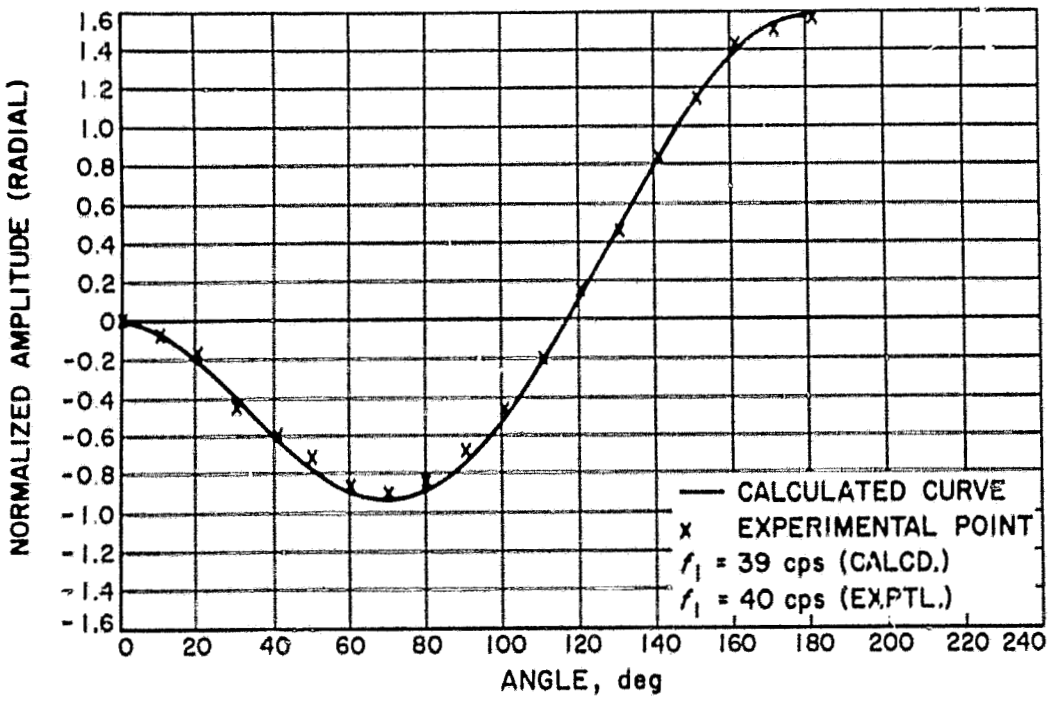

Fig. A-3. Mode shape of the first symmetric flexural mode of the "point-clamped" ring

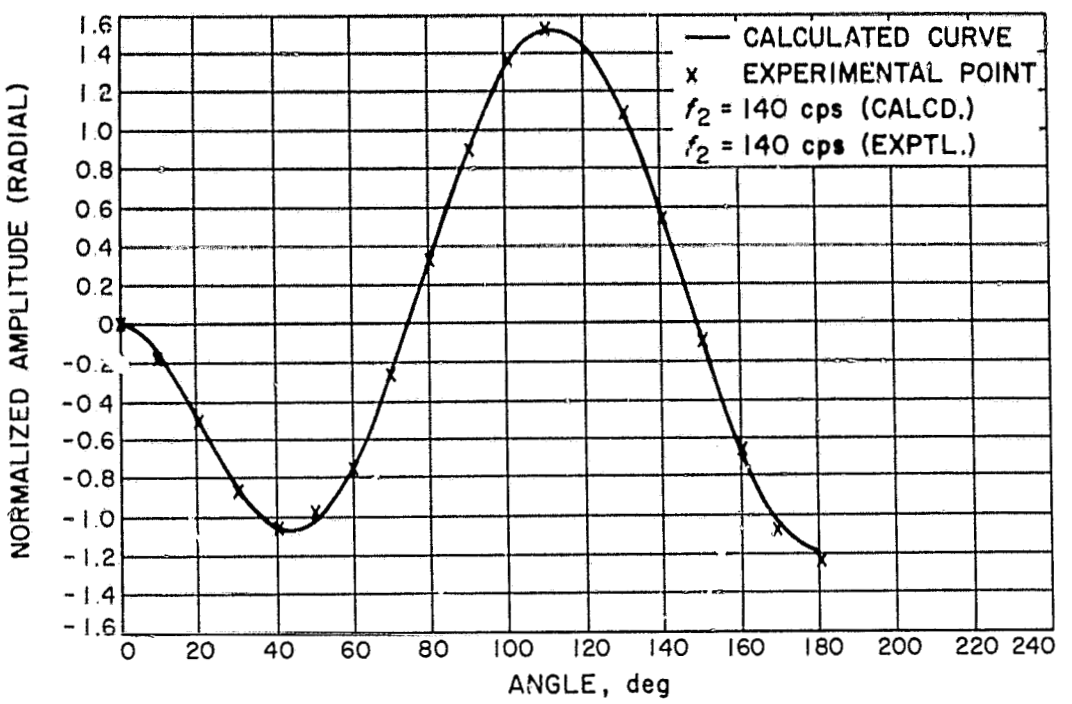

Fig. A-4. Mode shape of the second symmetric flexural mode of the "point-clamped" ring

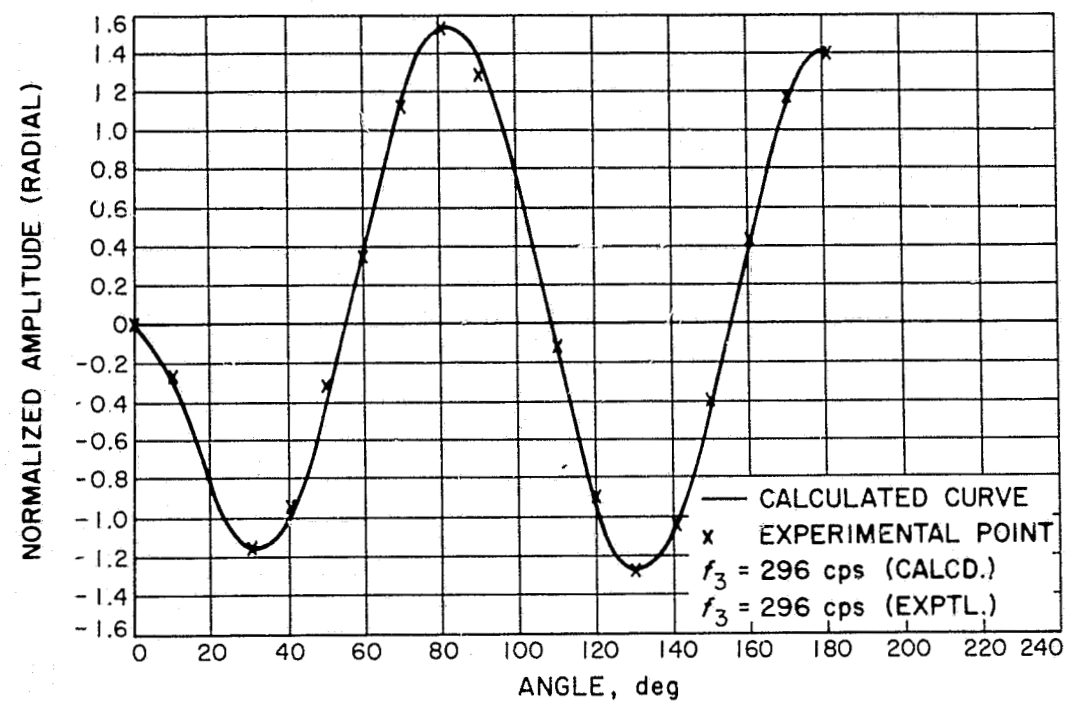

Fig. A-5. Mode shape of the third symmetric flexural mode of the "point-clamped" ring

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